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AN ANALYSIS OF STOCHASTIC DUELS
INVOLVING FIXED RATES OF FIRE

by

David Wiley Anderson

Thesis Advisor: J. G. Taylor

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An Analysis of Stochastic Duels
Involving Fixed Rates of Fire

by

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Submitted in partial fulfillment of the
requirements for the degree of

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ABSTRACT

This thesis presents an analysis of stochastic duels involving two opposing weapon systems with constant rates of fire. The duel was developed as a stationary Markov chain with stochastic matrices of transition probabilities constructed from the single shot kill probabilities of the weapon systems. A comparison was made of the presented Markov chain analysis results with results from other accepted conditional probability methods. As expected, this comparison established the validity of the Markov chain analysis and indicated advantages of the Markov chain approach in analysis of discrete process stochastic duels. The analysis was then extended to the two versus one duel where the three weapon systems were assumed to have fixed rates of fire.

TABLE OF CONTENTS

I. INTRODUCTION -- 4

- A. BACKGROUND -- 4
- B. PREVIOUSLY DEVELOPED ANALYSIS ------------------ 5
- C. APPLICATIONS OF STOCHASTIC DUELS -------------- 9

II. STOCHASTIC DUEL - FIXED RATES OF FIRE ------------- 11

- A. DEVELOPMENT OF MARKOV CHAIN MODEL ------------- 11
- B. EQUIVALENCE TO PREVIOUSLY DEVELOPED ANALYSIS -- 20
- C. EXAMPLES -- 22
- D. CHARACTERISTICS OF THIS MODEL ------------------ 27

III. TWO VERSUS ONE DUEL - FIXED RATES OF FIRE --------- 29

- A. DEVELOPMENT OF THE MODEL ----------------------- 29
- B. EXAMPLE --- 49

IV. CONCLUSIONS AND RECOMMENDATIONS ------------------ 51

- A. CONCLUSIONS ------------------------------------- 51
- B. RECOMMENDATIONS --------------------------------- 52

APPENDIX A: Some Convergence Proofs -------------------- 53

BIBLIOGRAPHY --- 55

INITIAL DISTRIBUTION LIST ------------------------------ 56

FORM DD 1473 --- 57

I. INTRODUCTION

A. BACKGROUND

The theory of stochastic duels has been developed in order to evaluate the probability of a weapon system surviving an engagement with another weapon system. The effect of weapon system parameters, such as single shot kill probabilities, rate of fire, and tactical (time) advantage can be determined in the analysis of these duels.

Joseph J. Schoderbeck (1962) and Trevor Williams and C. J. Ancker, Jr. (1963) developed the basic theory of the stochastic duel and the "fundamental duel" was defined as follows:

1. Two combatants, A and B, fired at each other until one was killed.
2. The time between rounds fired was either a constant or a random variable of known but different density function for each combatant.
3. Each combatant had a different known but fixed single shot kill probability.
4. The duel began with each combatant having unlimited ammunition supplies.
5. Both combatants had unlimited time in which to score a kill.
6. A tactical (time) advantage in firing the first shot was assigned to one of the combatants.

Since the first development of stochastic duel theory in 1962, many extensions to the theory have evolved. Analysis of weapon system duels constrained by ammunition limits, time limits, or varying single shot kill probabilities has been accomplished. Also, distributions of the time to kill and of the number of rounds fired have been determined. In addition two-versus-one duels and two-versus-two duels have been developed. As more combatants participate in the duel, it was shown that the results were in keeping with Lanchester models.

The objective of this thesis was to analyze the "fundamental duel" involving fixed time between firings, using stationary Markov chains. As expected the Markov process analysis was shown to be equivalent to the conditional probability methods. But in the Markov analysis of the stochastic duel, there existed advantages over other analysis methods that are indicated in Section II. Finally, the Markov analysis was extended to the two-versus-one duel where all three weapon systems were assumed to have fixed rates of fire.

B. PREVIOUSLY DEVELOPED ANALYSIS

Stochastic duels have been analyzed utilizing conditional probability methods. Schoderbeck [Ref. 1] was first responsible for the analysis of the fundamental duel involving fixed time between firings. A summary of his analysis is presented below.

A (friendly force) and B (enemy force) each possessed a single weapon system with single shot kill probabilities p_A and p_B respectively.

Let a = time between A's firings.

b = time between B's firings.

It was assumed that at time, $t=0$, A fired his first shot at B and that sometime later, T, B returned fire. Also, it was assumed that $a=b$, that $T<a$, and that each firing was an independent event. Then the firing sequence looked like that depicted in Fig. 1.

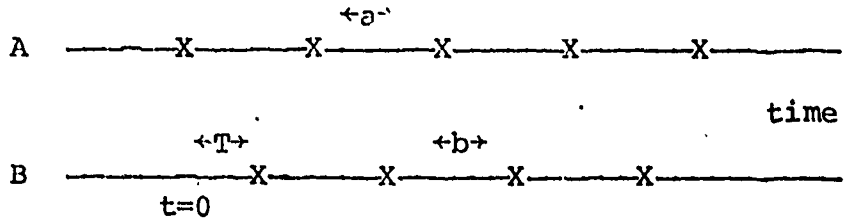

Figure 1. Firing Sequence of A and B.

Let $\Pr[A(t_o)]$ = Prob[A was alive and B was dead at time $t=t_o$] and

$$\Pr[A(\infty)] = \lim_{t_o \to \infty} \Pr[A(t_o)].$$

Considered first was the interval $a<t_o<a+T$. Within this interval there was the possibility that

a) A fired exactly two times.

b) B fired exactly one time.

Thus A was alive and B was dead if and only if

a) B was killed on A's first shot (so did not fire a shot), or

b) B missed with his one shot and was killed by A's second shot.

But, Pr[B was killed on A's first shot] = p_A

Pr[B missed with his first shot and was killed with A's second shot] = $(1-p_A)(1-p_B)p_A$ = $q_A q_B p_A$

and thus, $\Pr[A(t_o)] = p_A+q_Aq_Bp_A$ for $a<t_o<a+T$.

Then considered was the interval $a+T<t_o<2a+T$. In this interval there was the possibility that

a) A fired exactly three shots.

b) B fired exactly two shots.

And here A was alive and B was dead if and only if

a) B did not fire at all (killed by A's first shot),

or b) B fired once, missed, and was killed by A's second shot, or

c) B fired twice, missed twice, and was killed by A's third shot.

But, Pr[B did not fire at all] $= p_A$

Pr[B fired once, missed, and was killed by A's second shot]

$$= (1-p_A)(1-q_B)p_A = q_Aq_Bp_A$$

Pr[B fired twice, missed twice, and was killed by A's third shot] $= (1-p_A)(1-p_B)(1-p_A)(1-p_B)p_A = q_A^2q_B^2p_A$

Hence, $\Pr[A(t_o)] = p_A+q_Aq_Bp_A+q_A{}^2q_B{}^2p_A$ for $a+T<t_o<2a+T$

By induction then for the interval $(n-1)a+T<t_o<na+T$

$$\Pr[A(t_o)] = p_A+p_A(q_Aq_B)+p_A(q_Aq_B)^2+\cdots+p_A(q_Aq_B)^n$$

$$= p_A\left\{\frac{1-(q_Aq_B)^n}{1-q_Aq_B}\right\} \qquad \text{(I-1)}$$

and $$\Pr[A(\infty)] = \lim_{t_o\to\infty}\Pr[A(t_o)] = \lim_{n\to\infty} p_A\left\{\frac{1-(q_Aq_B)^n}{1-q_Aq_B}\right\}$$

$$= \frac{p_A}{1-q_Aq_B} \qquad \text{(I-2)}$$

Since $p_A > 0$ and $p_B > 0$ implied $q_A<1$ and $q_B<1$ and

$$\lim_{n\to\infty} (q_A q_B)^n = 0$$

Ancker and Williams [Ref. 2] developed a method for analyzing the fundamental duel with fixed firing rates, where $a \neq b$ but $T=0$. A summary of their analysis is presented below.

The ratio $\frac{a}{b}$ was assumed to be a rational number and if a and b contained a common factor, this ratio was reduced to α/β where α and β were relatively prime integers.

Let n = the largest number of times β was contained in α.

Let r = the remainder when β was divided into α, then $\alpha=n\beta+r$.

Let $P(A) = Pr(A \text{ won the duel}) = Pr(A \text{ was alive, } B \text{ was dead})$. Then A won on the j^{th} shot if he missed B on his first $j-1$ shots and had hit on his j^{th} round, while B missed with his first k rounds where $k=[j(\alpha/\beta)]$ and

$[X]$ = largest integer less than or equal to X.

Then the probability that A won the duel was

$$P(A) = \sum_{j=1}^{\infty} p_A q_A^{j-1} q_B^{k}$$

$$= p_A \sum_{j=1}^{\infty} q_A^{j-1} q_B^{jn+[j(r/\beta)]}$$

$$= \frac{p_A}{1-q_A^{\beta} q_B^{\alpha}} \sum_{j=0}^{\beta-1} q_A^{j} q_B^{[(j+1)\alpha/\beta]} \qquad \text{(I-3)}$$

C. APPLICATIONS OF STOCHASTIC DUELS

As analytical formulations of combat operations, stochastic duels are of great value in providing insight for the design of new weapon systems in advanced time frames. Stochastic duels lend themselves easily to parametric analysis. By varying parameters such as single shot kill probabilities or firing rates, it is possible using this analysis to determine tradeoffs between volume of fire versus accuracy of fire.

The military can apply the analysis of stochastic duels to missile sites, artillery, torpedoes, and most other weapon systems. Using realistic parameters, the survival probabilities can be determined in engagements with enemy weapon systems.

Analysis of duels involving fixed times between firings can also be compared with analysis of duels involving random times between firings, using the same parameter values. This comparison could yield useful information concerning the circumstances where a fixed rate of fire was more effective than a variable rate of fire.

Section II contains the presentation of the stationary Markov chain analysis of the fixed rate of fire stochastic duel. This method was verified also in Section II by comparing results with conditional probability analysis results which were presented in this Section. Examples of Markov analysis of some duels completed Section II.

Section III presented an extension of the analysis to the two-versus-one duel, including an example.

The conclusions reached after the analysis, and recommendations for further study are contained in Section IV.

II. STOCHASTIC DUEL – FIXED RATES OF FIRE

A. DEVELOPMENT OF THE MARKOV CHAIN MODEL

A and B represented two combatants, each possessing one weapon system. The weapon systems' respective single shot kill probabilities were p_A and p_B. Further, it was assumed that both A and B fired at a fixed rate so that a and b represented the respective times between firings (in seconds) of A and B. Then,

$$\lambda_A = \frac{60}{a} = \text{A's rate of fire (shots/min.)}$$

$$\lambda_B = \frac{60}{b} = \text{B's rate of fire (shots/min.)}$$

It was assumed also that both a and b were rational numbers and that A had an arbitrary tactical time advantage, T (seconds). Then:

A fired his first shot at time, t=0, and

B fired his first shot at time, t=T.

A, then, fired his j^{*th} shot at time

$$t_{Aj^*} = a(j^*-1) \qquad \text{(II-1)}$$

and B fired his k^{th} shot at time

$$t_{Bk} = T+b(k-1) \qquad \text{(II-2)}$$

illustrated in Fig. 2.

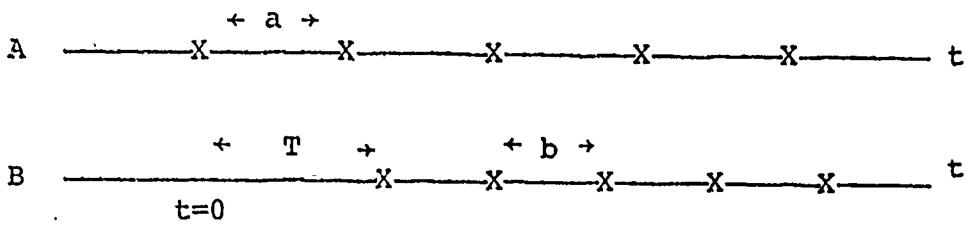

Figure 2. Time Relation Between A and B's Firings

At any time in the process, the duel was in one of four possible states:

STATE 1: A was alive; B was alive
STATE 2: A was alive; B was dead
STATE 3: A was dead; B was alive
STATE 4: A was dead; B was dead

It was noted that STATE 4 occurred only if A and B fired simultaneously. Whenever A or B (or both) fired, the duel underwent a transition from one state to another. Transition matrices associated with the three possible occurrences that could cause a transition from one state to another are in the form of the following:

1) A only fired

$$P_A = \begin{pmatrix} q_A & p_A & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

2) B only fired

$$P_B = \begin{pmatrix} q_B & 0 & p_B & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

3) A and B fired simultaneously

$$P_{AB} = \begin{pmatrix} q_Aq_B & p_Aq_B & q_Ap_B & p_Ap_B \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

where $q_A = 1-p_A$ and $q_B = 1-p_B$.

The initial state vector at the beginning of the duel was $\bar{P}_o=(1,0,0,0)$ since the duel was always in STATE 1 (A and B both alive) at the outset. After m transitions, however, the state vector was represented by

$$\bar{P}_m = \bar{P}_o \cdot P_1 \cdot P_2 \cdot \ldots \cdot P_m$$

where $P_i, i=1,2,\cdots,m$ was P_A, P_B, or P_{AB} depending on the firing sequence. This firing sequence was in turn dependent upon T, λ_A, and λ_B.

To compute $\bar{P}_m$, the number of times A fired in the period [0,T] first had to be determined. This was done as follows:

Set $t_{Aj*} = t_{B1}$, or

$$a(j^*-1) = T \text{ , and solved for } j^*$$

$j^* = \frac{T}{a} + 1$ and if $[j^*]$ indicated the greatest integer less than j^* , then $j = [j^*] = [\frac{T}{a} + 1]$ represented the number of times A fired in the period [0,T]. Then,

$$\bar{P}_m = \begin{cases} \bar{P}_o \cdot P_A{}^m & \text{for } m \leq j \\ \bar{P}_o \cdot P_A{}^j \cdot P_{j+1} \cdot P_{j+2} \cdot \ldots \cdot P_m & \text{for } j<m \end{cases} \qquad \text{(II-3)}$$

where P_i, $i = j + 1, j + 2, \cdots, m$ was P_A, P_B or P_{AB}

depending on firing sequence derived from λ_A and λ_B. Also beginning with P_{j+1} the sequences of matrices were periodic as shown below:

A fired his (j+1)st shot at time, $t \geq T$

B fired his 1st shot at time $t=T$

The time between B's 1st shot and A's (j+1)st shot was

$$t_{A,j+1} - t_{B1} = aj - T \qquad \text{(II-4)}$$

If τ represented the period in seconds of the firing sequence then the difference in time between the shot fired by A at time $t_{A,j+1} + n\tau$ and the shot fired by B at time $t_{B1} + n\tau$, $n=1,2,\cdots$, equaled the time interval of Eq. II-4. This was proven by the following method:

$\lambda_A = \frac{a_1}{a_2}$ and $\lambda_B = \frac{b_1}{b_2}$ where a_1, a_2, b_1, and b_2 were integers since a and b were assumed to be rational numbers.

Let L = lowest common multiple (L.C.M.) of a_2 and b_2

L = L.C.M. (a_2,b_2)

Then $\frac{L}{a_2}$ and $\frac{L}{b_2}$ were both integers.

$$L\lambda_A = a_1\left(\frac{L}{a_2}\right)$$

$$L\lambda_B = b_1\left(\frac{L}{b_2}\right)$$

Let L^* = L.C.M. $(L\lambda_A, L\lambda_B)$

Then $\tau = \frac{60\ L^*}{L\lambda_A\lambda_B}$ (seconds) was the period of the firing sequence. A, then, fired $\tau\left(\frac{\lambda_A}{60}\right) = \frac{\tau}{a}$ shots in τ seconds and B fired $\tau\left(\frac{\lambda_B}{60}\right) = \frac{\tau}{b}$ shots in τ seconds.

Let n_A = number of the shot A fired at time $t_{A,j+1} + n\tau$ and n_B = number of the shot B fired at time $t_{B1}+n\tau$.

$$n_A = j + 1 + \{n\tau(\frac{\lambda_A}{60})\}$$

$$n_B = 1 + \{n\tau(\frac{\lambda_B}{60})\} \text{ for } n = 1,2,\ldots$$

$$t_{A,n_A} = a(n_A-1) = \frac{60}{\lambda_A}\{j+n\tau(\frac{\lambda_A}{60})\} = \frac{60j}{\lambda_A} + n\tau$$

$$t_{B,n_B} = T + b(n_B-1) = T + \frac{60}{\lambda_B}\{n\tau(\frac{\lambda_B}{60})\} = T + n\tau$$

so that $t_{B,n_B} - t_{A,n_A} = \frac{60j}{\lambda_A} + n\tau - (T+n\tau) = \frac{60j}{\lambda_A} - T = a_j - T$

$$= t_{A,j+1} - t_{B1}$$

In a cycle then the total number of shots fired, N, was represented by $N = \tau(\frac{\lambda_A}{60}) + \tau(\frac{\lambda_B}{60})$ but $\tau(\frac{\lambda_A}{60}) = \frac{L^*}{L\lambda_B}$ and $\tau(\frac{\lambda_B}{60}) = \frac{L^*}{L\lambda_A}$ so that $N = \frac{L^*}{L}(\frac{1}{\lambda_B} + \frac{1}{\lambda_A})$.

If N* represented the maximum number of matrices per cycle,

then,

$$N^* = \begin{cases} N,\ (\frac{L^*}{L\lambda_B} \text{ of type } p_A\ ;\ \frac{L^*}{L\lambda_A} \text{ of type B) if there was no simultaneous firing} \\ N-1,\ ((\frac{L^*}{L\lambda_B}-1) \text{ of type } P_A,\ (\frac{L^*}{L\lambda_A} - 1) \text{ of type } P_B \text{ and 1 of type } P_{AB}) \text{ if there was a simultaneous firing.} \end{cases}$$

The state of the duel after n transition cycles (instead of m transitions), then, was written as:

$$\bar{P}_{j+n} = \bar{P}_j \cdot P_A{}^j \; (P_{j+1} \cdot P_{j+2} \cdot \ldots \cdot P_{j+m})^n$$

It was still necessary to determine if P_{j+i} for $i = 1,2,\cdots,m$ was a P_A, P_B, or P_{AB} matrix. This was determined as follows:

In the first of the n cycles A fired his (j+1)st, (j+2)nd, $\ldots, (j + \frac{L^*}{L\lambda_B})^{th}$ shots while B fired his 1st, 2nd,...., $(\frac{L^*}{L\lambda_A})^{th}$ shots and the time that each of these shots occurred was determined from Eqs. II-1 and II-2.

After ordering the matrices in the cycle let P* = product of these m matrices (m-1 if there were simultaneous firings in the sequence), i.e., $P^* = (P_{j+1} \cdot P_{j+2} \cdot \ldots \cdot P_{j+m})$. Also let $\bar{A}^* = \bar{P}_o \cdot P_A{}^j$. But since

$$P_A = \begin{pmatrix} q_A & p_A & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{then} \quad P_A{}^j = \begin{pmatrix} q_A{}^j & \left[p_A \sum_{i=1}^{j} q_A{}^{i-1}\right] & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So that $\bar{A}^* = \bar{P}_o \cdot P_A{}^j = (q_A{}^j \, , \, p_A \sum_{i=1}^{j} q_A{}^{i-1} \, , 0, 0) = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$

then $\bar{P}_{j+n} = \bar{A}^* P^{*n} = (P_{j+n,1}, P_{j+n,2}, P_{j+n,3}, P_{j+n,4})$.

For the solution to the duel, the individual terms of $\bar{P}_{j+n}$ were determined.

The matrix P* was of the form

$$P^* = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

then

$$P^{*2} = \begin{pmatrix} p_1^2 & p_2(1+p_1) & p_3(1+p_1) & p_4(1+p_1) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P^{*3} = \begin{pmatrix} p_1^3 & \left[p_2(1+p_1+p_1^2)\right] & \left[p_3(1+p_1+p_1^2)\right] & \left[p_4(1+p_1+p_1^2)\right] \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P^{*n} = \begin{pmatrix} p_1^n & \left[p_2\sum_{i=1}^{n} p_1^{i-1}\right] & \left[p_3\sum_{i=1}^{n} p_1^{i-1}\right] & \left[p_4\sum_{i=1}^{n} p_1^{i-1}\right] \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

It was apparent that the 2nd, 3rd and 4th terms of the first row of P^{*n} were geometric summations and were expressed as:

$$P^{*n} = \begin{pmatrix} p_1^n & p_2\left[\frac{1-p_1^n}{1-p_1}\right] & p_3\left[\frac{1-p_1^n}{1-p_1}\right] & p_4\left[\frac{1-p_1^n}{1-p_1}\right] \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence $\bar{P}_{j+n}$ was expressed as the following:

$$\bar{P}_{j+n} = \bar{A}^{*}P^{*n}$$

$$= \left(q_A^{\,j},\ p_A\sum_{i=1}^{j} q_A^{\,i-1},\ 0,\ 0\right)\begin{pmatrix} p_1^{\,n} & p_2\left[\frac{1-p_1^{\,n}}{1-p_1}\right] & p_3\left[\frac{1-p_1^{\,n}}{1-p_1}\right] & p_4\left[\frac{1-p_1^{\,n}}{1-p_1}\right] \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \left(p_1^{\,n}q_A^{\,j},\ p_2\left[\frac{1-p_1^{\,n}}{1-p_1}\right] q_A^{\,j} + p_A\sum_{i=1}^{j} q_A^{\,i-1},\ p_3\left[\frac{1-p_1^{\,n}}{1-p_1}\right] q_A^{\,j},\ p_4\left[\frac{1-p_1^{\,n}}{1-p_1}\right] q_A^{\,j}\right).$$

Since the objective of this analysis was the determination of the systems' survival probabilities as the number of transition cycles, n, got large, the results were presented in the form of Table I. As a quick check, it was easily seen that that the sum of the state probabilities in the limit as n→∞ was:

$$0 + p_A\sum_{i=1}^{j} q_A^{\,i-1} + q_A^{\,j}\left(\frac{p_2+p_3+p_4}{1-p_1}\right)$$

$$= p_A\left[\frac{1-q_A^{\,j}}{1-q_A}\right] + q_A^{\,j}\left(\frac{1-p_1}{1-p_1}\right)$$

$= 1 - q_A^{\,j} + q_A^{\,j} = 1$, as it should. Using the simplified notation where $\bar{A}^{*} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ then Table II was derived from Table I.

TABLE I

DUEL STATE PROBABILITIES

	After n Transition Cycles	As n → ∞
Pr (A alive; B alive)	$q_A^j \, p_1^n$	0
Pr (A alive; B dead)	$p_A \sum_{i=1}^{j} q_A^{i-1} + \frac{q_A^j \, p_2}{1-p_1} - \frac{q_A^j p_1^n p_2}{1-p_1}$	$p_A \sum_{i=1}^{j} q_A^{i-1} + \frac{q_A^j \, p_2}{1-p_1}$
Pr (A dead; B alive)	$\frac{q_A^j \, p_3}{1-p_1} - \frac{q_A^j \, p_1^n \, p_3}{1-p_1}$	$\frac{q_A^j \, p_3}{1-p_1}$
Pr (A dead; B dead)	$\frac{q_A^j \, p_4}{1-p_1} - \frac{q_A \, p_1^n \, p_4}{1-p_1}$	$\frac{q_A^j \, p_4}{1-p_1}$

TABLE II

DUEL STATE PROBABILITIES

	After n Transition Cycles	As n → ∞
Pr (A alive; B alive)	$\alpha_1 \, p_1^n$	0
Pr (A alive; B dead)	$\alpha_2 + \frac{\alpha_1 p_2}{1-p_1} - \frac{\alpha_1 p_1^n \, p_2}{1-p_1}$	$\alpha_2 + \frac{\alpha_1 \, p_2}{1-p_1}$
Pr (A dead; B alive	$\frac{\alpha_1 p_3}{1-p_1} - \frac{\alpha_1 p_1^n \, p_3}{1-p_1}$	$\frac{\alpha_1 \, p_3}{1-p_1}$
Pr (A dead; B dead)	$\frac{\alpha_1 p_4}{1-p_1} - \frac{\alpha_1 p_1^n \, p_4}{1-p_1}$	$\frac{\alpha_1 \, p_4}{1-p_1}$

B. EQUIVALENCE TO PREVIOUSLY DEVELOPED ANALYSIS

The MARKOV analysis of the Stochastic Duel involving fixed firing rates was shown to hold for the model developed by Schoderbeck and outlined in Section I. In this model, the times between firings, a and b, were assumed to be equal and the time advantage of combatant A was assumed to be less than a.

Given: $a = b$, $T<a$ $\quad \bar{P}_o = (1, 0, 0, 0)$

Then, $\lambda_A = \lambda_B$ and there was no possibility of a simultaneous firing at any time as the duel proceeded.

$$P_A = \begin{pmatrix} q_A & p_A & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad P_B = \begin{pmatrix} q_B & 0 & p_B & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$j = [\frac{T}{a} + 1] = 1 \text{ (since } 0<T<a)$$

$$\bar{A}^* = \bar{P}_o \cdot P_A = (q_A, p_A, 0, 0)$$

$$L = \text{L.C.M. } (a_2, b_2) = a_2$$

$$L^* = \text{L.C.M. } (L\lambda_A, L\lambda_B) = L\lambda_A$$

$$N = \frac{L^*}{L} (\frac{1}{\lambda_A} + \frac{1}{\lambda_B}) = \frac{2L^*}{L} (\frac{1}{\lambda_A}) = \frac{2L\lambda_A}{L\lambda_A} = 2$$

$$\frac{L^*}{L\lambda_B} = \frac{L\lambda_A}{L\lambda_B} = 1 \text{ shot by A}$$

$$\frac{L^*}{L\lambda_A} = \frac{L\lambda_A}{L\lambda_A} = 1 \text{ shot by B}$$

B fired 1st shot at time $t_{B,1} = T$

A fired 2nd shot at time $t_{A,2} = a$

Since T<a, cycle was of form BA, so P* was just the product of the two matrices, $P_A \cdot P_B$.

$$P^* = \begin{pmatrix} q_B & 0 & p_B & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} q_A & p_A & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} q_Aq_B & p_Aq_B & p_B & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$P^{*n} = \begin{pmatrix} (q_Aq_B)^n & p_Aq_B[\frac{1-(q_Aq_B)^n}{1-q_Aq_B}] & p_B[\frac{1-(q_Aq_B)^n}{1-q_Aq_B}] & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

$$\bar{P}_{1+n} = \bar{A}^*P^{*n} = (q_A(q_Aq_B)^n,\ p_A + p_Aq_Aq_B\ [\frac{1-(q_Aq_B)^n}{1-q_Aq_B}],$$

$$q_Ap_B[\frac{1-(q_Aq_B)^n}{1-q_Aq_B}],\ 0)$$

The results were best summarized in Table III. Checking the result obtained by this analysis and that developed in Section II, it was seen that the two methods were equivalent. But this was not unexpected, as the MARKOV chain analysis was basically the conditional probability analysis in matrix notation.

Analogously the MARKOV analysis was shown to yield equivalent results of the ANCKER-WILLIAMS model also summarized in Section I, as was expected. The clearest way to illustrate this equivalence was by example, since the firing rates were not specified in the ANCKER-WILLIAMS model, and examples are presented in the next Section.

TABLE III

DUEL STATE PROBABILITIES

	After n Transition Cycles	as n → ∞
Pr (A alive, B alive)	$q_A \, (q_A q_B)^n$	0
Pr (A alive, B dead)	$p_A \left\{\frac{1-(q_A q_B)^n}{1-q_A q_B}\right\}$	$\frac{p_A}{1-q_A q_B}$
Pr (A dead, B alive)	$q_A p_B \left\{\frac{1-(q_A q_B)^n}{1-q_A q_B}\right\}$	$\frac{q_A p_B}{1-q_A q_B}$
Pr (A dead, B dead)	0	0

C. EXAMPLES

In this section three examples are presented illustrating the use of the MARKOV analysis. The first two examples employed no time advantage for combatant A to show equivalence of this analysis to that developed by Ancker and Williams. The other example illustrated the type of duel that was able to be analyzed by the MARKOV method but not by either of the methods developed by Schoderbeck or Ancker and Williams.

EXAMPLE 1. $p_A = .6$, $p_B = .5$, $\lambda_A = 4$ shots per minute $\lambda_B = 2$ shots per minute, $T = 0$.

$$P_A = \begin{pmatrix} .4 & .6 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} P_B = \begin{pmatrix} .5 & 0 & .5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} P_{AB} = \begin{pmatrix} .2 & .3 & .2 & .3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\overline{P}_o = (1,\ 0,\ 0,\ 0)$$

$$\lambda_A = \frac{4}{1} = \frac{a_1}{a_2} \qquad \lambda_B = \frac{2}{1} = \frac{b_1}{b_2}$$

$$L = \text{L.C.M. } (a_2,\ b_2) = \text{L.C.M. } (1,\ 1) = 1$$

$$L\lambda_A = 4,\ L\lambda_B = 2$$

$$L^* = \text{L.C.M. } (L\lambda_A,\ L\lambda_B) = \text{L.C.M. } (4,\ 2) = 4$$

$$\tau = \frac{60\ L^*}{L\lambda_A\lambda_B} = \frac{(60)(4)}{(1)(4)(2)} = 30 \text{ sec.}$$

In 30 seconds A fired $\tau(\frac{\lambda_A}{60}) = \frac{(30)(4)}{60} = 2$ shots

and B fired $\tau(\frac{\lambda_B}{60}) = \frac{(30)(2)}{60} = 1$ shot

$$j = [\frac{T}{a} + 1] = [\frac{0}{15} + 1] = 0$$

$$N-1 = \tau(\frac{\lambda_A}{60} + \frac{\lambda_B}{60}) -1 = 30(\frac{4}{60} + \frac{2}{60}) -1 = 2$$

The number of P_A matrices $= (\frac{L^*}{L\lambda_B} - 1) = \frac{4}{(1)(2)} - 1 = 1$

and the number of P_B matrices $= (\frac{L^*}{L\lambda_A} - 1) = \frac{4}{(1)(4)} - 1 = 0$

and there was one P_{AB} matrix in the cycle.

Then,

$$\overline{A}^* = \overline{P}_o \cdot P_A{}^j = \overline{P}_o \cdot P_A{}^o = (1,\ 0,\ 0,\ 0) = (\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4)$$

$$P^* = P_A \cdot P_{AB} = \begin{pmatrix} .4 & .6 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} .2 & .3 & .2 & .3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} .08 & .72 & .08 & .12 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\overline{P}_j + n = \overline{P}_n = \overline{P}_o \cdot P^{*n}$$

So as $n \to \infty$, using results found in Table II,

$$\text{Pr (A alive; B dead)} = \alpha_2 + \frac{\alpha_1\ p_2}{1-p_1}$$

$$= 0 + \frac{(1)(.72)}{1-(.08)} = .78$$

Ancker and Williams method yielded the following result:

$$\text{Pr (A alive, B dead)} = \{\frac{p_A}{1-q_A^{\beta}\ q_B^{\alpha}}\} \sum_{j=0}^{\beta-1} q_A{}^{j}\ q_B{}^{[(j+1)-]}$$

$$a = \frac{60}{\lambda_A} = \frac{60}{4} = 15 \qquad b = \frac{60}{\lambda_B} = \frac{60}{2} = 30$$

$$\frac{a}{b} = \frac{15}{30} = \frac{1}{2} = \frac{\alpha}{\beta}$$

$$\text{Pr (A alive; B dead)}\ \{\frac{(.6)}{1-(.4)^2(.5)^1}\}\ [(.4)^0(.5)^0$$

$$+ (.4)^1(.5)^1] = .78$$

EXAMPLE 2: $p_A = .2$, $p_B = .8$, $\lambda_A = 5$ shots per minute,

$\lambda_B = 2$ shots per minute, T = 0

$$P_A = \begin{pmatrix} .8 & .2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad P_B = \begin{pmatrix} .2 & 0 & .8 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad P_{AB} = \begin{pmatrix} .16 & .04 & .64 & .16 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\overline{P}_o = (1, 0, 0, 0)$$

$$\lambda_A = \frac{5}{1} = \frac{a_1}{a_2} \qquad \lambda_B = \frac{2}{1} = \frac{b_1}{b_2}$$

$$L = \text{L.C.M. } (1,1) = 1, \quad L\lambda_A = 5,\ L\lambda_B = 2$$

$$L^* = \text{L.C.M. } (5.2) = 10$$

$$\tau = \frac{60\ L^*}{L\lambda_A\lambda_B} = \frac{(60)(10)}{(1)(5)(2)} = 60 \text{ sec.}$$

In 60 seconds A fired $\tau(\frac{\lambda_A}{60}) = 60(\frac{5}{60}) = 5$ shots, and

B fired $\tau(\frac{\lambda_B}{60}) = 60(\frac{2}{60}) = 2$ shots

$$j = [\frac{T}{a} + 1] = 0$$

$$N-1 = \tau(\frac{\lambda_A}{60} + \frac{\lambda_B}{60}) - 1 = 60(\frac{5}{60} + \frac{2}{60}) -1 = 6$$

The number of P_A matrices in a firing cycle $= (\frac{L^*}{L\lambda_B} - 1)$

$$= \frac{10}{(1)(2)} - 1 = 4, \text{ and}$$

the number of P_B matrices $= \frac{L^*}{L\lambda_A} - 1) = (\frac{10}{(1)(5)} - 1) = 1$

and there was one P_{AB} matrix.

Then,

$$\bar{A}^* = \bar{P}_o \cdot P_A{}^j = \bar{P}_o \cdot P_A{}^o = (1, 0, 0, 0) = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$$

$$P^* = P_A P_A P_B P_A P_A P_{AB}$$

and after doing this multiplication of matrices,

$$P^* = \begin{pmatrix} .013 & .410 & .564 & .013 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and as $n \to \infty$, using results from Table II.

$$\text{Pr (A alive; B dead)} = \alpha_2 + \frac{\alpha_1 p_2}{1-p_1}$$

$$= 0 + \frac{(1)(.410)}{1-.013} = .415$$

Ancker and Williams method yielded:

$$\text{Pr (A alive; B dead)} = .415$$

EXAMPLE 3: $p_A = .8$, $p_B = .5$, $\lambda_A = 4$ shots per minute, $\lambda_B = 3$ shots per minute, $T = 15$ seconds.

$$P_A = \begin{pmatrix} .2 & .8 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad P_B = \begin{pmatrix} .5 & 0 & .5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad P_{AB} = \begin{pmatrix} .1 & .4 & .1 & .4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\bar{P}_o = (1, 0, 0, 0)$$

$$\lambda_A = \frac{4}{1} = \frac{a_1}{a_2} \qquad \lambda_B = \frac{3}{1} = \frac{b_1}{b_2}$$

$$L = \text{L.C.M. } (1, 1) = 1, \quad L\lambda_A = 4, \quad L\lambda_B = 3$$

$$L^* = \text{L.C.M. } (4, 3) = 12$$

$$\tau = \frac{60\ L^*}{L\lambda_A\lambda_B} = \frac{(60)(12)}{(1)(4)(3)} = 60 \text{ seconds}$$

In 60 seconds A fired $\tau(\frac{\lambda_A}{60}) = 60\ (\frac{4}{60}) = 4$ shots, and

B fired $\tau(\frac{\lambda_B}{60}) = 60\ (\frac{3}{60}) = 3$ shots

$$j = [\frac{T}{a}+1] = [\frac{15}{15}+1] = 1$$

$$N-1 = \tau(\frac{\lambda_A}{60} + \frac{\lambda_B}{60}) - 1 = 4 + 3 - 1 = 6$$

The number of P_A matrices in a firing cycle $= (\frac{L^*}{L\lambda_B} - 1)$

$$= \frac{12}{(1)(3)} - 1 = 3$$

and the number of P_B matrices =

$$(\frac{L^*}{L\lambda_A} - 1) = \frac{12}{(1)(4)} - 1 = 2$$

and there was one P_{AB} matrix. Then,

$$\bar{A}^* = \bar{P}_0 \cdot P_A{}^j = \bar{P}_o \cdot P_A = (.2, .8, 0, 0) = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$$

$$P^* = P_{AB}P_AP_BP_AP_BP_A$$

This matrix multiplication yielded:

$$P^* = \begin{pmatrix} .0002 & .4888 & .1110 & .4000 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and as $n \to \infty$, again results were taken from Table II.

$$\Pr \text{(A alive, B dead)} = \alpha_2 + \frac{\alpha_1 p_2}{1-p_1}$$

$$= .8 + \frac{(.2)(.4888)}{1-.0002} = .8980$$

D. CHARACTERISTICS OF THIS MODEL

The MARKOV chain analysis of the Stochastic Duel was only another form of the conditional probability models presented in Section I. However, the Markov chain model enabled the analyst to consider duels where one side had a positive time advantage with no restrictions on the length of the time advantage. The model developed here allowed the analysis of duels involving different but fixed rates of fire by the two combatants.

In contrast, Schoderbeck's model of the fixed firing rate duel was restricted to a time advantage less than the time period between two rounds from one combatant and further restricted to the case where each combatant had the same firing rate. Ancker and Williams' development of the fixed firing rate duel was restricted to the case where neither combatant had a time advantage and both started firing simultaneously. Also their model only furnished results for the outcome probabilities in the limiting case after an infinite number of exchanges. The Markov model yielded results for the limiting case and the case after a finite number of transition cycles, hence a finite number of rounds.

Another feature of this model was the ease in which it could be computer programmed. Analysts could be interested in the duel outcome probabilities if either or both combatants had limited ammunition. For given firing rates the number of transition cycles before one combatant's ammunition was exhausted could be computed. Then Pr (A alive; B dead) could be determined at that point.

Possible extensions of this analysis along with uses of the fixed firing rate stochastic duel are presented in Section IV.

III. TWO VERSUS ONE DUEL—FIXED RATES OF FIRE

A. DEVELOPMENT OF THE MODEL

In the model combatant A dueled with two other combatants, B and C. The same assumptions as in the one-on-one duel held, but here A was assumed to have two weapon systems and fired at B with one and at C with the other.

Single shot kill probabilities, then were:

p_{AB} = Pr (A killed B) (on one shot at B)
p_{AC} = Pr (A killed C)
p_{BA} = Pr (B killed A)
p_{CA} = Pr (C killed A)
p_{BC} = Pr (B killed C) = 0
p_{CB} = Pr (C killed B) = 0

Firing rates were:

λ_{AB} = rate of A's fire at B (shots per minute)
λ_{AC} = rate of A's fire at C (" " ")
λ_{BA} = rate of B's fire at A (" " ")
λ_{CA} = rate of C's fire at A (" " ")

At any time in the process, then, the duel was in one of eight possible states:

STATE 1: A alive; B alive; C alive
STATE 2: A alive; B alive; C dead
STATE 3: A alive; B dead; C alive
STATE 4: A alive; B dead; C dead
STATE 5: A dead; B alive; C alive
STATE 6: A dead; B alive; C dead
STATE 7: A dead; B dead; C alive
STATE 8: A dead; B dead; C dead

Transition matrices associated with the fifteen possible occurrences that could cause a transition from one state to another are presented below:

CASE 1: A fired at B

$$P_{AB} = \begin{pmatrix} q_{AB} & 0 & p_{AB} & 0 & 0 & 0 & 0 & 0 \\ 0 & q_{AB} & 0 & p_{AB} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

CASE 2: A fired at C

$$P_{AC} = \begin{pmatrix} q_{AC} & p_{AC} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{AC} & p_{AC} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

CASE 3: B fired at A

$$P_{BA} = \begin{pmatrix} q_{BA} & 0 & 0 & 0 & p_{BA} & 0 & 0 & 0 \\ 0 & q_{BA} & 0 & 0 & 0 & p_{BA} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

CASE 4: C fired at A

$$P_{CA} = \begin{pmatrix} q_{CA} & 0 & 0 & 0 & p_{CA} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{CA} & 0 & 0 & 0 & p_{CA} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

CASE 5: A fired at B; B fired at A

$$P_{AB\cdot BA} = \begin{pmatrix} [q_{AB}\, q_{BA}] & 0 & [p_{AB}\, q_{BA}] & 0 & [q_{AB}\, p_{BA}] & 0 & [p_{AB}\, p_{BA}] & 0 \\ 0 & [q_{AB}\, q_{BA}] & 0 & [p_{AB}\, q_{BA}] & 0 & [q_{AB}\, p_{BA}] & 0 & [p_{AB}\, p_{BA}] \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

CASE 6: A fired at B; C fired at A

$$p_{AB\cdot CA} = \begin{pmatrix} [q_{AB}\ q_{CA}] & 0 & [p_{AB}\ q_{CA}] & 0 & [q_{AB}\ p_{CA}] & 0 & [p_{AB}\ p_{CA}] & 0 \\ 0 & q_{AB} & 0 & p_{AB} & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{CA} & 0 & 0 & 0 & p_{CA} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

CASE 7: A fired at C; B fired at A

$$p_{AC\cdot BA} = \begin{pmatrix} [q_{AC}\ q_{BA}] & [p_{AC}\ q_{BA}] & 0 & 0 & [q_{AC}\ p_{BA}] & [p_{AC}\ p_{BA}] & 0 & 0 \\ 0 & q_{BA} & 0 & 0 & 0 & p_{BA} & 0 & 0 \\ 0 & 0 & q_{AC} & p_{AC} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

CASE 8: A fired at C; C fired at A

$$p_{AC\cdot CA} = \begin{pmatrix} [q_{AC}q_{CA}] & [p_{AC}q_{CA}] & 0 & 0 & [q_{AC}p_{CA}] & [p_{AC}p_{CA}] & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & [q_{AC}q_{CA}] & [p_{AC}q_{CA}] & 0 & 0 & [q_{AC}p_{CA}] & [p_{AC}p_{CA}] \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

CASE 9: A fired at B; A fired at C

$$P_{AB\cdot AC} = \begin{pmatrix} [q_{AB}\ q_{AC}] & [q_{AB}\ p_{AC}] & [p_{AB}\ q_{AC}] & [p_{AB}\ p_{AC}] & 0 & 0 & 0 & 0 \\ 0 & q_{AB} & 0 & p_{AB} & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{AC} & p_{AC} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

CASE 10: B fired at A; C fired at A

$$P_{BA\cdot CA} = \begin{pmatrix} [q_{BA}\ q_{CA}] & 0 & 0 & 0 & 0 & [p_{BA}q_{CA}+q_{BA}p_{CA}+p_{BA}p_{CA}] & 0 & 0 \\ 0 & q_{BA} & 0 & 0 & 0 & p_{BA} & 0 & 0 \\ 0 & 0 & q_{CA} & 0 & 0 & 0 & p_{CA} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

CASE 11: A fired at B; A fired at C; B fired at A

$$P_{AB \cdot AC \cdot BA} = \begin{pmatrix} [q_{AB}q_{AC}q_{BA}] & [q_{AB}p_{AC}q_{BA}] & [p_{AB}q_{AC}q_{BA}] & [p_{AB}p_{AC}q_{BA}] & [q_{AB}q_{AC}p_{BA}] & [q_{AB}p_{AC}p_{BA}] & [p_{AB}q_{AC}p_{BA}] & [p_{AB}p_{AC}p_{BA}] \\ 0 & [q_{AB}q_{BA}] & 0 & [p_{AB}q_{BA}] & 0 & [q_{AB}p_{BA}] & 0 & [p_{AB}p_{BA}] \\ 0 & 0 & q_{AC} & p_{AC} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

CASE 12: A fired at B; A fired at C; C fired at A

$$P_{AB.AC.CA} = \left(\begin{array}{cccccccc}
[q_{AB}q_{AC}q_{CA}] & [q_{AB}p_{AC}q_{CA}] & [p_{AB}q_{AC}q_{CA}] & [p_{AB}p_{AC}q_{CA}] & [q_{AB}q_{AC}p_{CA}] & [q_{AB}p_{AC}p_{CA}] & [p_{AB}q_{AC}p_{CA}] & [p_{AB}p_{AC}p_{CA}] \\
0 & q_{AB} & 0 & p_{AB} & 0 & 0 & 0 & 0 \\
0 & 0 & [q_{AC}q_{CA}] & [p_{AC}q_{CA}] & 0 & 0 & [q_{AC}p_{CA}] & [p_{AC}p_{CA}] \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right)$$

CASE 13: A fired at B; B fired at A; C fired at A

$$P_{AB.BA.CA} = \begin{pmatrix} \left[q_{AC}q_{BA}q_{CA}\right] & 0 & \left[p_{AB}q_{BA}q_{CA}\right] & 0 & \left[\begin{matrix} q_{AB}(p_{BA}q_{CA}+ \\ q_{BA}p_{CA}+p_{BA}p_{CA}) \end{matrix}\right] & 0 & \left[\begin{matrix} p_{AB}(p_{BA}q_{CA}+ \\ q_{BA}p_{CA}+p_{BA}p_{CA}) \end{matrix}\right] & 0 \\ 0 & \left[q_{AB}q_{BA}\right] & 0 & \left[p_{AB}q_{BA}\right] & 0 & \left[q_{AB}p_{BA}\right] & 0 & \left[p_{AB}p_{BA}\right] \\ 0 & 0 & q_{CA} & 0 & 0 & 0 & p_{CA} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

CASE 14: A fired at C; B fired at A; C fired at A

$$P_{AC.BA.CA} = \begin{pmatrix} \left[q_{AC}q_{BA}q_{CA}\right] & \left[p_{AC}q_{BA}q_{CA}\right] & 0 & 0 & \begin{bmatrix} q_{AC}(p_{BA}q_{CA}+ \\ q_{BA}p_{CA}+p_{BA}p_{CA}) \end{bmatrix} & \begin{bmatrix} p_{AC}(p_{BA}q_{CA}+ \\ q_{BA}p_{CA}+p_{BA}p_{CA}) \end{bmatrix} & 0 & 0 \\ 0 & q_{BA} & 0 & 0 & 0 & p_{BA} & 0 & 0 \\ 0 & 0 & \left[q_{AC}q_{CA}\right] & \left[p_{AC}q_{CA}\right] & 0 & 0 & \left[q_{AC}p_{CA}\right] & \left[p_{AC}p_{CA}\right] \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

CASE 15: A fired at B; A fired at C; B fired at A; C fired at A

$$P_{\substack{AB.AC.\\BA.CA}} = \begin{pmatrix}
\begin{bmatrix}(q_{AB}q_{AC}\cdot\\ q_{BA}q_{CA})\end{bmatrix} & \begin{bmatrix}(q_{AB}p_{AC}\cdot\\ q_{BA}q_{CA})\end{bmatrix} & \begin{bmatrix}(p_{AB}q_{AC}\cdot\\ q_{BA}q_{CA})\end{bmatrix} & \begin{bmatrix}(p_{AB}p_{AC}\cdot\\ q_{BA}q_{CA})\end{bmatrix} & \begin{bmatrix}q_{AB}q_{AC}\ \times\\ (q_{BA}p_{CA}+\\ p_{BA}q_{CA}\ +\\ p_{BA}p_{CA})\end{bmatrix} & \begin{bmatrix}q_{AB}p_{AC}\ \times\\ (q_{BA}p_{CA}+\\ p_{BA}q_{CA}+\\ p_{BA}p_{CA})\end{bmatrix} & \begin{bmatrix}p_{AB}q_{AC}\ \times\\ (q_{BA}p_{CA}+\\ p_{BA}q_{CA}+\\ p_{BA}p_{CA})\end{bmatrix} & \begin{bmatrix}p_{AB}p_{AC}\ \times\\ (q_{BA}p_{CA}+\\ p_{BA}q_{CA}\ +\\ p_{BA}p_{CA})\end{bmatrix} \\
0 & [q_{AB}q_{BA}] & 0 & [p_{AB}q_{BA}] & 0 & [q_{AB}p_{BA}] & 0 & [p_{AB}p_{BA}] \\
0 & 0 & [q_{AC}q_{CA}] & [p_{AC}q_{CA}] & 0 & 0 & [q_{AC}p_{CA}] & [p_{AC}p_{CA}] \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

In this model the initial state vector was:

$$\bar{P}_o = (1, 0, 0, 0, 0, 0, 0, 0)$$

It was further assumed here that A had time advantages over both combatants B and C and that these time advantages were equal. In order to proceed with the analysis, it was then necessary to compute the number of times A fired at each of the other combatants.

Let T = time advantage A had over B and C (in seconds). Then, the number of times A fired at B in the interval [0, T] was determined as follows:

$$\text{Set } t_{AB,jB} = t_{BA,1} \text{ or}$$

$$\frac{60}{\lambda_{AB}}(jB^* - 1) = T$$

$$jB^* = \frac{\lambda_{AB}T}{60} + 1$$

Then, $jB = [jB^*]$ represented the number of times A fired at B in the interval [0, T], where $[\cdot]$ was as defined in Section II. Likewise $jC = [jC^*] = [\frac{\lambda_{AC}T}{60} + 1]$ represented the number of times A fired at C in the interval [0,T].

Since by the nature of the matrices P_{AB} and P_{AC}, multiplication of these two matrices in any order was commutative so that $\bar{A}^* = \bar{P}_o \cdot P_{AB}{}^{jB} \cdot P_{AC}{}^{jC}$. Performing this matrix multiplication, it was seen that the form of $P_{AB}{}^{jB}$ was:

$$p_{AB}^{jB} = \begin{pmatrix} q_{AB}^{jB} & 0 & \left[p_{AB}\sum_{i=1}^{jB} q_{AB}^{i-1}\right] & 0 & 0 & 0 & 0 & 0 \\ 0 & q_{AB}^{jB} & 0 & \left[p_{AB}\sum_{i=1}^{jB} q_{AB}^{i-1}\right] & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Likewise the form of p_{AC}^{jC} was:

$$p_{AC}^{jC} = \begin{pmatrix} q_{AC}^{jC} & \left[p_{AC}\sum_{i=1}^{jC} q_{AC}^{i-1}\right] & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{AC}^{jC} & \left[p_{AC}\sum_{i=1}^{jC} q_{AC}^{i-1}\right] & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

It followed that $p_{AB}^{jB} p_{AC}^{jC} = p_{AC}^{jC} p_{AB}^{jB}$ and was of the form:

$$p_{AB}^{jB} p_{AC}^{jC} = \begin{pmatrix} \left[q_{AB}^{jB} q_{AC}^{jC}\right] & \left[q_{AB}^{jB} p_{AC} \sum_{i=1}^{jC} q_{AC}^{i-1}\right] & \left[p_{AB} q_{AC}^{jC} \sum_{i=1}^{jB} q_{AB}^{i-1}\right] & \left[p_{AB} p_{AC} \sum_{i=1}^{jB} q_{AB}^{i-1} \sum_{i=1}^{jC} q_{AC}^{i-1}\right] & 0 & 0 & 0 & 0 \\ 0 & q_{AB}^{jB} & 0 & \left[p_{AB} \sum_{i=1}^{jB} q_{AB}^{i=1}\right] & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{AC}^{jC} & \left[p_{AC} \sum_{i=1}^{jC} q_{AC}^{i=1}\right] & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and, then

$$\bar{A}^* = \left(q_{AB}^{\,jB}\, q_{AC}^{\,jC},\ q_{AB}^{\,jB}\, p_{AC} \sum_{i=1}^{jC} q_{AC}^{\,i-1},\ p_{AB}\, q_{AC}^{\,jC} \sum_{i=1}^{jB} q_{AB}^{\,i-1},\ p_{AB}\, p_{AC} \sum_{i=1}^{jB} q_{AB}^{\,i-1} \sum_{i=1}^{jC} q_{AC}^{\,i-1},\ 0,\ 0,\ 0,\ 0\right)$$

$$= (\alpha_1, \alpha_2, \alpha_3, \alpha_4, 0, 0, 0, 0).$$

It followed from the derivation in Section II that the firing sequence in the two-versus-one duel was also periodic. The combination of the fifteen possible transition matrices depended on the firing sequence, which was again dependent on λ_{AB}, λ_{AC}, λ_{BA}, λ_{CA}, and T. In any case the form of P*, the matrix representing one firing cycle was of the form:

$$P^* = P_{j+1} \cdot P_{j+2} \cdot \ldots \cdot P_{j+M} \quad \text{where } j = jB + jC$$

$$P^* = \begin{pmatrix} p_{11} & p_{12} & p_{13} & p_{14} & p_{15} & p_{16} & p_{17} & p_{18} \\ 0 & p_{22} & 0 & p_{24} & 0 & p_{26} & 0 & p_{28} \\ 0 & 0 & p_{33} & p_{34} & 0 & 0 & p_{37} & p_{38} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

After n of these transition cycles, then:

$$P^{*n} = \left(\begin{array}{ccc}
[p_{11}{}^{n}] & [p_{12}\sum_{i=1}^{n} p_{11}{}^{i-1} p_{22}{}^{n-i}] & [p_{13}\sum_{i=1}^{n} p_{11}{}^{i-1} p_{33}{}^{n-i}] \\
0 & p_{22}{}^{n} & 0 \\
0 & 0 & p_{33} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\underline{\text{Col. 1}} & \underline{\text{Col. 2}} & \underline{\text{Col. 3}}
\end{array} \right.$$

$$\begin{array}{cc}
\left[\begin{array}{l} p_{14}\sum_{i=1}^{n} p_{11}{}^{i-1} \\ + \; p_{12}p_{24}\sum_{j=0}^{n-2} \; \sum_{i=0}^{j} p_{11}{}^{i} p_{22}{}^{j-i} \\ + \; p_{13}p_{34}\sum_{j=0}^{n-2} \; \sum_{i=0}^{j} p_{11}{}^{i} p_{33}{}^{j-i} \end{array} \right] & [p_{15}\sum_{i=1}^{n} p_{11}{}^{i-1}] \\
\left[p_{24}\sum_{i=1}^{n} p_{22}{}^{i-1} \right] & 0 \\
\left[p_{34}\sum_{i=1}^{n} p_{33}{}^{i-1} \right] & 0 \\
1 & 0 \\
0 & 1 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
\underline{\text{Col. 4}} & \underline{\text{Col. 5}}
\end{array}$$

$$\begin{array}{c}
\left[\begin{array}{c} p_{16}\sum\limits_{i=1}^{n} p_{11}^{\,i-1} \\ + p_{12}p_{26}\sum\limits_{j=0}^{n-2}\ \sum\limits_{i=0}^{j} p_{11}^{\,i} p_{22}^{\,j-i} \end{array}\right] \\
p_{26}\sum\limits_{i=1}^{n} p_{22}^{\,i-1} \\
0 \\
0 \\
0 \\
1 \\
0 \\
0
\end{array}
\qquad
\begin{array}{c}
\left[\begin{array}{c} p_{17}\sum\limits_{i=1}^{n} p_{11}^{\,i-1} \\ + p_{13}p_{37}\sum\limits_{j=0}^{n-2}\ \sum\limits_{i=0}^{j} p_{11}^{\,i}\ p_{33}^{\,j-i} \end{array}\right] \\
0 \\
\left[p_{37}\sum\limits_{i=1}^{n} p_{33}^{\,i-1}\right] \\
0 \\
0 \\
0 \\
1 \\
0
\end{array}$$

Col. 6 Col. 7

$$\left(\begin{array}{c}
\left[\begin{array}{c} p_{18}\sum\limits_{i=1}^{n} p_{11}^{\,i-1} \\ + p_{12}p_{28}\sum\limits_{j=0}^{n-2}\ \sum\limits_{i=0}^{j} p_{11}^{\,i} p_{22}^{\,j-i} \\ + p_{13}p_{38}\sum\limits_{j=0}^{n-2}\ \sum\limits_{i=0}^{j} p_{11}^{\,i} p_{33}^{\,j-i} \end{array}\right] \\
\left[p_{28}\sum\limits_{i=1}^{n} p_{22}^{\,i-1}\right] \\
\left[p_{38}\sum\limits_{i=1}^{n} p_{33}^{\,i-1}\right] \\
0 \\
0 \\
0 \\
0 \\
1
\end{array}\right)$$

Col. 8

The probability that the duel was in any one of the eight possible states after n transition cycles was taken from:

$$\bar{P}_{j+n} = \bar{A}^{*} P^{*n}$$

$$= (\Pr(\text{STATE } 1), \Pr(\text{STATE } 2), \ldots, \Pr(\text{STATE } 8))$$

and the results of this multiplication is presented in Table IV.

Then, utilizing the notation for $\bar{A}^{*}$, i.e.

$$\bar{A}^{*} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, 0, 0, 0, 0)$$

the state probabilities of the duel were computed as the number of transition cycles got very large ($n \to \infty$). Table V was then derived from Table IV.

TABLE IV

DUEL STATE PROBABILITIES

State Probabilities	After n Transition Cycles
Pr (STATE 1)	$q_{AB}^{jB}\, q_{AC}^{jC}\, p_{11}^{n}$
Pr (STATE 2)	$q_{AB}^{jB}\, q_{AC}^{jC}\, p_{12} \sum_{i=1}^{n} p_{11}^{i-1} p_{22}^{n-i}$
	$+\ p_{AC}\, q_{AB}^{jC}\, p_{22}^{n} \sum_{i=1}^{jC} q_{AC}^{i-1}$
Pr (STATE 3)	$q_{AB}^{jB}\, q_{AC}^{jC}\, p_{13} \sum_{i=1}^{n} p_{11}^{i-1} p_{33}^{n-i}$
	$+\ p_{AB}\, q_{AC}^{jC}\, p_{13}^{n} \sum_{i=1}^{jB} q_{AB}^{i-1}$
Pr (STATE 4)	$q_{AB}^{jB}\, q_{AC}^{jC} [p_{14} \sum_{i=1}^{n} p_{11}^{i-1}$
	$+\ p_{12}\, p_{24} \sum_{j=0}^{n-2} \sum_{i=0}^{j} p_{11}^{i} p_{22}^{j-i}$
	$+\ p_{13}\, p_{34} \sum_{j=0}^{n-1} \sum_{i=0}^{j} p_{11}^{i} p_{33}^{j-i}]$
	$+\ p_{AC} q_{AB}^{jB} p_{24} \sum_{i=1}^{jC} q_{AC}^{i-1} \sum_{i=1}^{n} p_{22}^{i-1}$
	$+\ p_{AB} q_{AC}^{jC} p_{34} \sum_{i=1}^{jB} q_{AB}^{i-1} \sum_{i=1}^{n} p_{33}^{i-1}$
	$+\ p_{AB} p_{AC} \sum_{i=1}^{jB} q_{AB}^{i-1} \sum_{i=1}^{jC} q_{AC}^{i-1}$

TABLE IV--Continued

State Probabilities	After n Transition Cycles
Pr (STATE 5)	$q_{AB}^{jB}\ q_{AC}^{jC}\ p_{15}\sum_{i=1}^{n} p_{11}^{i-1}$
Pr (STATE 6)	$q_{AB}^{jB}\ q_{AC}^{jC}[p_{16}\sum_{i=1}^{n} p_{11}^{i-1}$
	$+\ p_{12}p_{26}\sum_{j=0}^{n-2}\ \sum_{i=0}^{j} p_{11}^{i}p_{22}^{j-i}]$
	$+\ p_{AC}\ q_{AB}^{jB}\ p_{26}\sum_{i=1}^{jC} q_{AC}^{i-1}\ \sum_{i=1}^{n} p_{22}^{i-1}$
Pr (STATE 7)	$q_{AB}^{jB}\ q_{AC}^{jC}[p_{17}\sum_{i=1}^{n} p_{11}^{i-1}$
	$+\ p_{13}p_{37}\sum_{j=0}^{n-1}\ \sum_{i=0}^{j} p_{11}^{i}p_{33}^{j-i}]$
	$+\ p_{AB}\ q_{AC}^{jC}\ p_{37}\sum_{i=1}^{jB} q_{AB}^{i-1}\ \sum_{i=1}^{n} p_{33}^{i-1}$
Pr (STATE 8)	$q_{AB}^{jB}\ q_{AC}^{jC}[p_{18}\sum_{i=1}^{n} p_{11}^{i-1}$
	$+\ p_{12}\ p_{18}\sum_{j=0}^{n-2}\ \sum_{i=0}^{j} p_{11}^{i}p_{22}^{j-i}$
	$+\ p_{13}\ p_{38}\sum_{j=0}^{n-1}\ \sum_{i=0}^{j} p_{11}^{i}p_{33}^{j-i}]$
	$+\ p_{AC}\ q_{AB}^{jB}\ p_{28}\sum_{i=1}^{jC} q_{AC}^{i-1}\ \sum_{i=1}^{n} p_{22}^{i-1}$
	$+\ p_{AB}\ q_{AC}^{jC}\ p_{38}\sum_{i=1}^{jB} q_{AB}^{i-1}\ \sum_{i=1}^{n} p_{33}^{i-1}$

TABLE V

DUEL STATE PROBABILITIES

State Probabilities	As n → ∞
Pr (STATE 1)	0
Pr (STATE 2)	0
Pr (STATE 3)	0
Pr (STATE 4)	$\frac{\alpha_1 p_{14}}{1-p_{11}} + \frac{\alpha_1 p_{12} p_{24} p_{12}^*}{(1-p_{12}^*)^2} + \frac{\alpha_1 p_{13} p_{34} p_{13}^*}{(1-p_{13}^*)^2} + \frac{\alpha_2 p_{24}}{1-p_{22}} + \frac{\alpha_3 p_{34}}{1-p_{33}} + \alpha_4$
Pr (STATE 5)	$\frac{\alpha_1 p_{15}}{1-p_{11}}$
Pr (STATE 6)	$\frac{\alpha_1 p_{16}}{1-p_{11}} + \frac{\alpha_1 p_{12} p_{26} p_{12}^*}{(1-p_{12}^*)^2} + \frac{\alpha_2 p_{26}}{1-p_{22}}$
Pr (STATE 7)	$\frac{\alpha_1 p_{17}}{1-p_{11}} + \frac{\alpha_1 p_{13} p_{37} p_{13}^*}{(1-p_{13}^*)^2} + \frac{\alpha_3 p_{37}}{1-p_{33}}$
Pr (STATE 8)	$\frac{\alpha_1 p_{18}}{1-p_{11}} + \frac{\alpha_1 p_{12} p_{28} p_{12}^*}{(1-p_{12}^*)^2} + \frac{\alpha_1 p_{13} p_{38} p_{13}^*}{(1-p_{13}^*)^2} + \frac{\alpha_2 p_{28}}{1-p_{22}} + \frac{\alpha_3 p_{38}}{1-p_{33}}$

Where $p_{12}^* = \max(p_{11}, p_{22})$ and $p_{13}^* = \max(p_{11}, p_{33})$

To see how the summations of the form

$$\sum_{i=1}^{n} p_{11}^{\,i-1} p_{kk}^{\,n-i} \quad \text{and} \quad \sum_{j=0}^{n-2} \sum_{i=0}^{j} p_{11}^{\,i}\, p_{kk}^{\,j-i}, \quad k = 2, 3$$

converged see Appendix A.

EXAMPLE:

Let $p_{AB} = .8$, $p_{AC} = .8$, $p_{BA} = .6$, $p_{CA} = .5$

$\lambda_{AB} = 4$ shots per minute, $\lambda_{AC} = 2$ shots per minute

$\lambda_{BA} = 3$ shots per minute, $\lambda_{CA} = 2$ shots per minute

and T = 15 sec. The firing sequence is illustrated in Fig. 3.

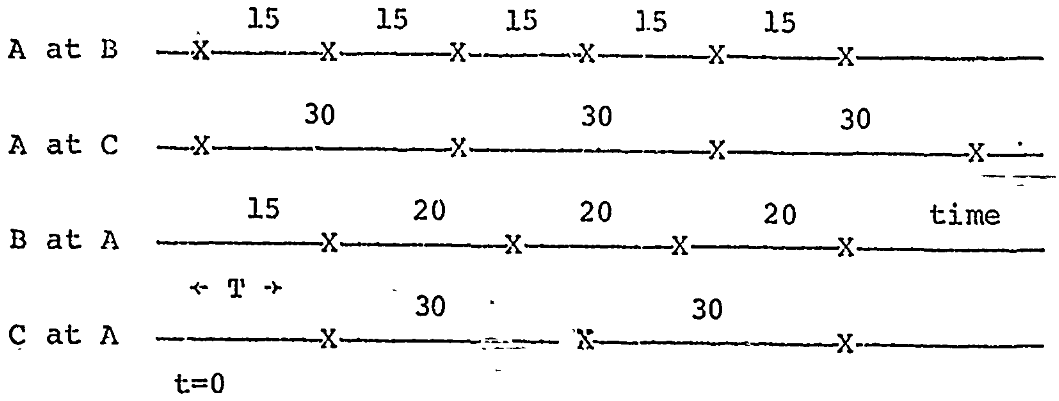

Figure 3. Firing Sequence

Then $\bar{P}_o = (1, 0, 0, 0, 0, 0, 0, 0)$

$$jB = [\frac{\lambda_{AB}T}{60} + 1] = [\frac{(4)(15)}{60} + 1] = 1$$

$$jC = [\frac{\lambda_{AC}T}{60} + 1] = [\frac{(2)(15)}{60} + 1] = 1$$

$$A^* = \bar{P}_o \, P_{AB}^{jB} \, P_{AC}^{jC} = \bar{P}_o \cdot P_{AB} \, P_{AC}$$

$$= (.04, .16, .16, .64, 0, 0, 0, 0)$$

$$P^* = P_{2+1} \cdot P_{2+2} \cdot \ldots P_{2+6}$$

$$= P_{AB \cdot BA \cdot CA} \, P_{AB \cdot AC} \, P_{BA} \, P_{AB \cdot CA} \, P_{BA} \, P_{AB \cdot AC}$$

Multiplying these matrices yielded:

$$P^*=\begin{pmatrix} .000001 & .000045 & .003896 & .171393 & .161062 & .004147 & .659456 & 0 \\ 0 & .000102 & 0 & .388861 & 0 & .131037 & 0 & .48 \\ 0 & 0 & .01 & .44 & 0 & 0 & .55 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Thus, at end of duel (as $n \to \infty$)

Pr (A alive; B dead; C dead) = Pr (STATE 4)

$$= \frac{\alpha_1 p_{14}}{1-p_{11}} + \frac{\alpha_1 p_{12} p_{24} p_{12}^*}{(1-p_{12}^*)^2} + \frac{\alpha_1 p_{13} p_{34} p_{13}^*}{(1-p_{13}^*)^2} + \frac{\alpha_2 p_{24}}{1-p_{22}} + \frac{\alpha_3 p_{34}}{1-p_{33}} + \alpha_4$$

where $p_{12}^* = \max(p_{11}, p_{22}) = .000102$ and

$p_{13}^* = \max(p_{11}, p_{33}) = .01$

Pr (STATE 4) = .77758

IV. CONCLUSIONS AND RECOMMENDATIONS

A. CONCLUSIONS

The development in this thesis of the fixed firing rate stochastic duel was presented to serve as an analytical tool in evaluating present or proposed weapon systems. The use of Markov chains in the model enabled the analysis of more complex but also more realistic weapon system engagements than other models previously developed. The state probabilities of the model in the limit as the number of transition cycles increased were functions only of single shot kill probabilities and hence easily computable and computer programmable. By computer programming this model, parametric studies of firing rates, single shot kill probabilities and time advantage could be performed. Also, if the number of transition cycles was fixed as a function of one combatant's limited ammunition supply, then state probabilities can be determined for the duel where ammunition is limited.

The two-versus-one duel evolved utilizing the same technique as the fundamental duel analysis. Results were more complex, however state probabilities still depended solely on kill probabilities and time advantage. It was thought that the two-versus-one duel would be extremely difficult to develop as a conditional probability model, even without considering a time advantage.

Stochastic duels involving combatants with fixed rates of fire can be of considerable importance in the evaluation of weapon systems. Using models developed by C. J. Ancker and others, the fixed rate of fire duel can be compared to the duel where time between firing is a random variable. Analysis of the random firing rate and fixed firing rate models could yield the optimal firing doctrine for given parameters. To do this, the mean values of the random firing rates should be set equal to the fixed firing rates in the Markov chain model.

B. RECOMMENDATIONS

Further analysis into the fixed rate of fire stochastic duel can consider the following areas:

1. Analysis of the duel considering properties inherent to discrete parameter Markov chains found in Parzen [Ref. 4] and specifically investigating such properties as:
    a. First passage probabilities and first passage times.
    b. Absorption probabilities and mean absorption times.
    c. Stationary distributions.
    d. Limiting occupation times.
2. Analysis of the two-versus-one duel where the time advantage combatant A has over B and that which A has over C are unequal.
3. The distribution of rounds fired in a stochastic duel utilizing techniques of C. J. Ancker, Jr., and A. V. Gafarian [Ref. 3].

APPENDIX A

SOME CONVERGENCE PROOFS

A-I. $$\lim_{n\to\infty} \sum_{i=1}^{n} p_{11}^{\,i-1}\, p_{kk}^{\,n-i} = 0; \quad k = 2, 3; \; p_{11}, p_{kk} \quad 1$$

PROOF: Let $p_{11} \geq p_{kk}$ and substitute p_{11} for p_{kk} in the expression above. This yields:

$$\lim_{n\to\infty} \sum_{i=1}^{n} p_{11}^{\,n-1} = \lim_{n\to\infty} n\, p_{11}^{\,n-1}$$

Applying L' Hospital's rule:

$$\lim_{n\to\infty} \left(\frac{n}{p_{11}^{\,-n+1}}\right) = \lim_{n\to\infty} \left[\frac{1}{-p_{11}^{\,-n+1} \ln p_{11}}\right] = 0$$

A-II. $$\lim_{n\to\infty} \sum_{j=0}^{n-2} \sum_{i=0}^{j} p_{11}^{\,i}\, p_{kk}^{\,j-i} = \frac{p_{1k}^{*}}{(1-p_{1k}^{*})^{2}};$$

$$k = 2, 3; \; p_{11}, p_{kk} < 1$$

where $p_{1k}^{*} = \max\,(p_{11}, p_{kk})$

PROOF: Substitute p_{1k}^{*} for p_{11} and p_{kk} in above expression. This yields:

$$\lim_{n\to\infty} \sum_{j=0}^{n-2} \sum_{i=0}^{j} p_{1k}^{*j} = \lim_{n\to\infty} \sum_{j=0}^{n-2} j\, p_{1k}^{*j}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} p_{1k}{}^{*} \sum_{j=0}^{n-2} \frac{d}{dp_{1k}{}^{*}} (p_{1k}{}^{*j}) \\
&= \lim_{n\to\infty} p_{1k}{}^{*} \frac{d}{dp_{1k}{}^{*}} \sum_{j=0}^{n-2} p_{1k}{}^{*j} \\
&= \lim_{n\to\infty} p_{1k}{}^{*} \frac{d}{dp_{1k}{}^{*}} \left[\frac{1-p_{1k}{}^{*n-1}}{1-p_{1k}{}^{*}}\right] \\
&= \lim_{n\to\infty} p_{1k}{}^{*} \left[\frac{(1-p_{1k}{}^{*})[-(n-1)p_{1k}{}^{*n-2}]-(1-p_{1k}{}^{*n-1})(-1)}{(1-p_{1k}{}^{*})^{2}}\right] \\
&= \lim_{n\to\infty} \left\{\frac{p_{1k}{}^{*}}{(1-p_{1k}{}^{*})^{2}} + \frac{p_{1k}{}^{*}}{(1-p_{1k}{}^{*})^{2}} [(n-2)p_{1k}{}^{*n-1}] \right. \\
&\qquad \left. - \frac{p_{1k}{}^{*}}{(1-p_{1k}{}^{*})^{2}} [(n-1)p_{1k}{}^{*n-2}]\right\}
\end{aligned}
\tag{A-II-1}
$$

but the second and third terms of Eq. A-II-1 went to zero in the limit as n→∞ by the proof of A-I above.

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13. ABSTRACT

This thesis presents an analysis of stochastic duels involving two opposing weapon systems with constant rates of fire. The duel was developed as a stationary Markov chain with stochastic matrices of transition probabilities constructed from the single shot kill probabilities of the weapon systems. A comparison was made of the presented Markov chain analysis results with results from other accepted conditional probability methods. As expected, this comparison established the validity of the Markov chain analysis and indicated advantages of the Markov chain approach in analysis of discrete process stochastic duels. The analysis was then extended to the two versus one duel where the three weapon systems were assumed to have fixed rates of fire.

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